ON ROULETTE WHEN THE HOLES ARE OF VARIOUS SIZES

BY

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ABSTRACT

For the usual goal problems, even if the holes on a roulette table are of various sizes and give different odds, there is no advantage in placing a positive stake on more than one hole on any particular spin.

This note considers a roulette table in which the holes are of various sizes and the odds vary from hole to hole. The question arises as to how someone with an infinitely divisible fortune should play this kind of roulette so as to maximize the probability of ultimately reaching a specified larger fortune. It is shown that *there is no advantage in ever placing a bet on more than one hole on any single spin.* However, only in exceptional cases would it be wise to use the same hole throughout the entire course of play.

Actually a little more is contained in the theorem below, for it handles the more general case in which several—possibily infinitely many—roulette tables are simultaneously available. But the possibility of simultaneously betting on infinitely many holes is not included.

In the interesting special case of ordinary roulette in which the odds and the sizes are reciprocals of integers and do not vary from hole to hole, the italicised results above are due to Smith [4]. At least in this case, there is indeed a genuine disadvantage in always placing a stake on more than one hole, as will be explained below.

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What is a roulette table; that is, what gambles does it make available? To each hole in a roulette table is associated two parameters w, r, $0 < w, r < 1$, where w represents the probability that the ball will fall in the hole and r represents the ratio of the amount staked on that hole to the amount returned for betting on that hole if the ball falls in that hole. Thus, for ordinary roulette, r may be taken as $1/36$ and, if the table is in Monte Carlo, w may be taken as $1/37$.

For the purposes of this note, any family of pairs of numbers (w, r) with $0 < w$, $r < 1$, say (w_α, r_α), determines a *roulette-table* $\hat{\Gamma}$, and α can be identified with the hole corresponding to (w_α, r_α). If a gamble γ available in $\hat{\Gamma}$ stakes a positive amount on only one hole, say $s_\alpha r_\alpha$ on hole α, then the fortune either decreases by $s_\alpha r_\alpha$ or increases by $s_\alpha(1 - r_\alpha)$ with probabilities $1 - w_\alpha$ and w_α respectively.

For gambles γ that stake positive amounts on more than one hole it is convenient *to adopt the convention that Σ_γ signifies summation over all the holes on which γ stakes a positive amount.*

Using this convention, in order that γ be a gamble it is necessary that $\Sigma_\gamma w_\alpha \leqq 1$. If γ stakes a positive amount on each of several holes say $s_\alpha r_\alpha$ on hole α, then the fortune decreases by $\Sigma_\gamma s_\alpha r_\alpha$ with probability $1 - \Sigma_\gamma w_\alpha$ and increases by $s_\beta - \Sigma_\gamma s_\alpha r_\alpha$ with probability w_β, for each β for which $s_\beta > 0$; if the initial fortune ts f, and the ball falls in hole β, then the new fortune f_β is given by

$$f_\beta = f + s_\beta - \Sigma_\gamma s_\alpha r_\alpha. \tag{1}$$

For γ to be available, it is necessary that $f_\beta \geqq 0$ for all β; here s_β in (1) may be 0. Moreover, in this note, γ is available in $\hat{\Gamma}$ only if the set of α on which γ places a positive stake is finite. As is easily verified, with or without this condition, the house $\hat{\Gamma}$ thus defined is a casino in the technical sense of [3, chap. 4]. (Possibly Theorem 1 holds for the larger $\hat{\Gamma}$ too.)

To each (w, r) correspond a primitive casino (w, r), which is indistinguishable from a roulette-table with only one hole, and is formally defined in chap. 6 of [3].

Let Γ be the union of all $\Gamma(w, r)$ as (w, r) ranges over the set of (w_α, r_α). As is evident, $\Gamma \subset \hat{\Gamma}$, so the U of Γ and $\hat{\Gamma}$, say U and $\hat{U}$, plainly satisfy $U \leqq \hat{U}$.

The *usual* u is the indicator function of the closed half infinite interval $[1, \infty)$.

The result of this note can now be formulated thus.

THEOREM I. *For the usual u, the U of the union of primitive casinos is the U of the corresponding roulette-table, that is, $U = \hat{U}$.*

To prove Theorem 1, it suffices to show that U is excessive for $\hat{\Gamma}$, as [3, th. 2.12.1] makes plain. That is, what must be shown is that the availability of γ in $\hat{\Gamma}$ at f necessitates $\gamma U \leqq U(f)$.

Of course, the U of any house is excessive for that house, as is established in [3, th. 2.14.1]. Therefore, to prove Theorem 1, it is sufficient to establish:

PROPOSITION 1. *Any casino function Q that is excessive for a set of primitive casinos Γ_α is excessive for the corresponding roulette-table.*

Set aside the superfair case in which some w_α exceeds r_α—for this case is easily handled by the general theory of [3, chap. 4]—and restrict attention henceforth to the more interesting case in which $w_\alpha \leqq r_\alpha$ for each α:

A gamble γ that places positive stakes on precisely n holes is at *order n*. The program will be to establish:

LEMMA 1. *Let $0 \leqq f \leqq 1$, let $1 \leqq n$, and let γ available in $\hat{\Gamma}$ at f, be of order $n+1$. Then there is a γ^* available there of order at most n such that, for all casino functions Q that are excessive for Γ,*

$$\gamma Q \leqq \gamma^* Q. \tag{2}$$

That Lemma 1 suffices to establish Proposition 1 is easily seen. For, by induction, there would necessarily be a γ^* available in $\hat{\Gamma}$ at f that places a positive stake on at most one hole, and for which (2) holds. But such a γ^* is available in Γ at f. And, since Q is excessive for Γ,

$$\gamma^* Q \leqq Q(f). \tag{3}$$

Plainly, (2) together with (3) would yield the desired conclusion $\gamma Q \leqq Q(f)$.

Turn now to the proof of Lemma 1. Suppose that the least positive value of s_α is assumed when $\alpha = \delta$. Let γ^* stake 0 on every hole on which γ stakes 0, and also 0 on hole δ. On each remaining hole α, let γ^* stake $s_\alpha^* r_\alpha$, where

$$s_\alpha^* = s_\alpha - cs\delta; \tag{4}$$

and where $c = 1$ if $\Sigma_\gamma r_\alpha \geqq 1$, and, otherwise,

$$c = \frac{r_\delta}{1 - \Sigma_\gamma r_\alpha + r_\delta}. \tag{5}$$

As is easily seen, γ^* is available in $\hat{\Gamma}$ at f, and γ^* has an order less than the order of γ. Moreover, if $\Sigma_\gamma r_\alpha \geqq 1$, then $f_\beta^* \geqq f_\beta$ for all β, where f_β is as in (1), as is easily verified. Thus, γ^* dominates γ, and (2) holds for every nondecreasing Q.

The remainder of this note is devoted to showing that (2) holds also when $\Sigma_\gamma r_\alpha < 1$. As is easily verified by elementary algebra, the gambles γ^* and γ are then related by

(6) $f^*_\alpha = f_\alpha$ for every α for which $s_\alpha > 0$, except for $\alpha = \delta$.

Of course, (6) greatly facilitates the computations needed to verify (2). The successful association of such a gamble γ^* to γ is the crucial idea that Smith introduced in [4].

In calculating $\gamma^*Q - \gamma Q$, all terms cancel except three: (i) a total loss under γ^*, (ii) a total loss under γ, and (iii) the outcome under γ happens be δ. Call the corresponding fortunes f_0, f_0 and f_δ. Then

$$\gamma^*Q - \gamma Q = (1 - \Sigma_{\gamma^*}w_\alpha)Q(f^*_0) - (1 - \Sigma_\gamma w_\alpha)Q(f_0) - w_\delta Q(f_\delta). \tag{7}$$

Since

$$\begin{aligned} f^*_0 &= f - \Sigma_{\gamma^*}s^*_\alpha r_\alpha, \\ f_0 &= f - \Sigma_\gamma s_\alpha r_\alpha, \end{aligned} \tag{8}$$

and

$$f = f_0 + s_\delta,$$

it is simple to verify that

$$f^*_0 = cf_\delta + (1 - c)f_0. \tag{9}$$

Dividing both sides of (7) by the coefficient of $Q(f^*_0)$, one sees that

$$\gamma^*Q - \gamma Q \geqq 0 \quad \text{if, and only if,}$$

$$Q(cf_\delta + (1 - c)f_0) \geqq WQ(f_\delta) + (1 - W)Q(f_0) \tag{10}$$

where

$$W = \frac{w_\delta}{1 - \Sigma_\gamma w_\alpha + w_\delta}. \tag{11}$$

All in all, the problem that now remains is to verify (10). By the general casino inequality [3, chap. 4], one sees that the left side of (10) is at least as large as $Q(c)Q(f_\delta) + (1 - Q(c))Q(f_0)$. Therefore, to verify (10), it certainly suffices that

$$Q(c) \geqq W, \tag{12}$$

where c is defined by (5) and W by (11).

Recapitulating, what is needed, therefore, is this:

SUBLEMMA 1. *Suppose* $0 < w_\alpha < r_\alpha < 1$, $\Sigma r_\alpha < 1$, *and* Q *is a subfair casino function that is excessive for the primitive casinos determined by the* (w_α, r_α). *Then, for all* δ,

$$Q\left(\frac{r_\delta}{1-\Sigma r_\alpha + r_\delta}\right) \geqq \frac{w_\delta}{1-\Sigma w_\alpha + w_\delta}. \tag{13}$$

Since $Q(r_\alpha) \geqq w_\alpha$ for all α, as [3, th. 2.21.1] implies,

$$\frac{Q(r_\delta)}{1-\Sigma Q(r_\alpha) + Q(r_\delta)} \geqq \frac{w_\delta}{1-\Sigma w_\alpha + w_\delta}. \tag{14}$$

Therefore, what remains to be shown is that the left side of (13) majorizes the left side of (14). And this is a special case of:

A casino inequality

For any subfair casino function Q, *(and even for any bonnded, nontrivial* Q *that satisfies the two special casino inequalities in* [3])

$$Q\left(\frac{f}{1-g}\right) \geqq \frac{Q(f)}{1-Q(g)}, \tag{15}$$

for $0 < f < f + g \leqq 1$;

and, more generaily,

$$Q\left(\frac{f_1}{1-\Sigma_{j\geqq 2} f_j}\right) \geqq \frac{Q(f_1)}{1-\Sigma_{j\geqq 2} Q(f_\delta)}, \tag{16}$$

for $0 < f_1 \leqq \Sigma_{j\geqq 1} f_j \leqq 1$, *where, for each* j, $f_j > 0$, *and* j *runs over a finite or a denumerable set of indices.*

PROOF OF THE INEQUALITY. Compute thus:

$$\begin{aligned} Q\left(\frac{f}{1-g}\right) &= Q(f + fg + fg^2 + \cdots) \\ &\geqq Q(f) + Q(fg) + Q(fg^2) + \cdots \\ &\geqq Q(f) + Q(f)Q(g) + Q(f)Q^2(g) + \cdots \\ &= Q(f)[1 + Q(g) + Q^2(g) + \cdots] \\ &= \frac{Q(f)}{1-Q(g)}, \end{aligned} \tag{17}$$

where the first inequality uses the superadditivity of Q, as in [1], and the second uses a casino inequality in [3, chap. 4].

For (16), compute thus:

$$Q\left(\frac{f_1}{1-\Sigma_{j\geqq 2}f_j}\right) \geqq \frac{Q(f_1)}{1-Q(\Sigma_{j\geqq 2}f_j)} \geqq \frac{Q(f_1)}{1-\Sigma_{j\geqq 2}Q(f_j)}, \tag{18}$$

where the first inequality is an instance of (15), and the second is another expression of the superadditivity of Q. This completes the proof of Theorem 1.

That *multiple-hole bets γ are strictly disadvantageous for ordinary roulette* is now an easy consequence of the strict superadditivity of subfair primitive casino functions (3, th. 6.5.1). For if Q is such a function, the inequality in (15) is strict, which, tracing through the proof of Theorem 1, yields $\gamma Q < Q(f)$ for the appropriate casino function Q.

COROLLARY. *The U of the roulette table $\hat{\Gamma}$ is the infimum of all casino functions that majorize each of the corresponding primitive, casino functions.*

PROOF. The infimum Q is a casino function (3, th. 4.6.1). Plainly, $Q \leqq U_0$, the U of the union Γ of the primitive casinos associated, with $\hat{\Gamma}$. In view of (3, th. 6.7.2), Q is excessive for Γ. So, as implied by (3, th. 2.12.1), $Q \geqq U_0$. Hence, $Q = U_0$, and $U_0 = U$, by Theorem 1.

Only in the exceptional case in which one of the primitive casino functions associated with the roulette table majorizes all the others would it be optimal to bet on the same hole for all initial fortunes.

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